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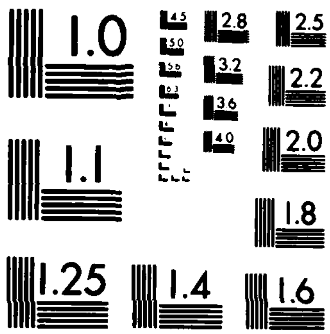

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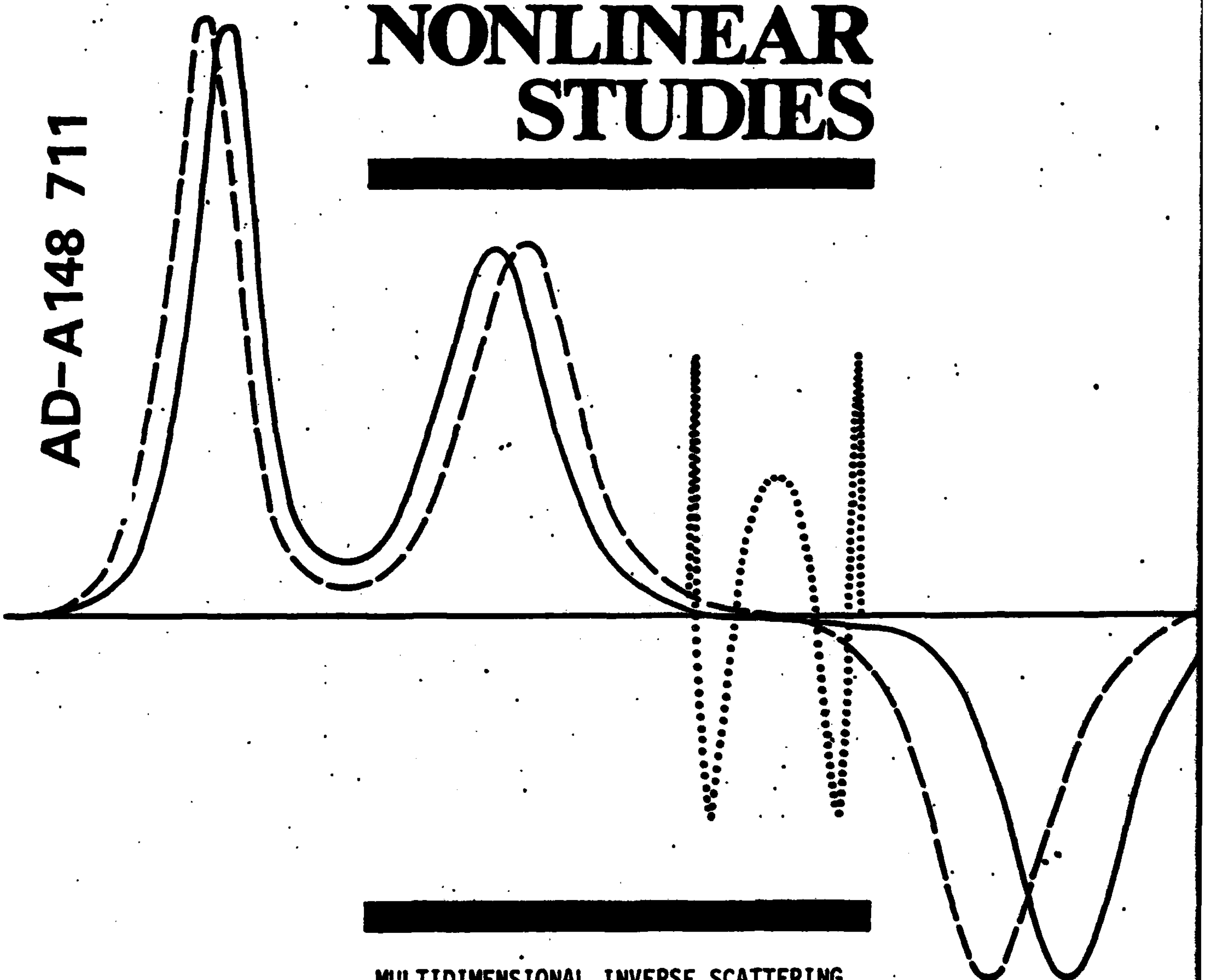

MULTIDIMENSIONAL INVERSE SCATTERING FOR FIRST ORDER SYSTEMS

by

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ABSTRACT

A method for solving the inverse problem for a class of multidimensional first order systems is given. The analysis yields equations which the scattering data must satisfy; these equations are natural candidates for characterizing admissible scattering data. The results are used to solve the multidimensional N-wave resonant interaction equations.

1. Introduction

The inverse scattering problems for the hyperbolic and elliptic generalizations in the plane of the m x m AKNS system have been successfully studied in [1] and applied to the linearization of several physically significant nonlinear evolution equations (N-wave interaction, Davey-Stewartson, etc.) in two spatial and one temporal dimensions.

We indicate here how the method used in our investigation of the n-dimensional Schrödinger equation [2] can be applied to the study of the inverse problem for the operator in $\mathbb{R}^{n+1}$:

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$$L_\sigma \doteq \frac{\partial}{\partial x_0} + \sigma \sum_{\ell=1}^{n} J_\ell \frac{\partial}{\partial x_\ell} - Q(x_0,x) \ . \tag{1}$$

Here J_ℓ are constant real diagonal m x m matrices (we denote the diagonal entries of J_ℓ by $J_\ell^1,\dots,J_\ell^m$ and assume $J_\ell^i \neq J_\ell^j \neq 0$ whenever $i \neq j$); the matrix-valued off-diagonal potential $Q = (Q^{ij})$ may depend on x_0 as well as $x = (x_1,\dots,x_n)$ and $\sigma = \sigma_R + i\sigma_I$ is a complex parameter.

The operator (1) is associated with the nonlinear system:

$$\frac{\partial Q^{ij}}{\partial t} = \frac{1}{\sigma} a_{ij} \frac{\partial Q^{ij}}{\partial x_0} + \sum_\ell (a_{ij}J_\ell^i - B_\ell^i) \frac{\partial Q^{ij}}{\partial x_\ell} + \frac{1}{\sigma} \sum_\ell (a_{i\ell}-a_{\ell j})Q^{i\ell}Q^{\ell j} \tag{2}$$

$$\left(\text{with } a_{ij} = \frac{B_\ell^j - B_\ell^i}{J_\ell^j - J_\ell^i}, \quad 1 \leqq \ell \leqq n, \quad \text{for some real } B_\ell^i \ \ 1 \leqq \ell \leqq n, \ \ 1 \leqq i \leqq m\right). \tag{3}$$

Even though no traditional scattering operator exists in the case $\sigma_I \neq 0$, the so-called $\bar\partial$ method (see [2] and references given there) gives a satisfactory definition of scattering data for L_σ, along with a systematic inversion procedure, which we use to solve (2).

A solution of the inverse scattering problem for the hyperbolic case $\sigma_I = 0$ is then obtained by a limiting argument; thus we can avoid a separate study of a Riemann-Hilbert boundary value problem (which in higher dimensions may also involve some geometric complications). The main advantage of this approach is that it yields (from the compatibility conditions associated with $\bar\partial$ in several variables) equations which must be satisfied by the scattering data. In addition to their importance for the problem of characterizing admissible scattering data, these equations have several consequences: i) they provide a formula for reconstructing the potential from the scattering transform purely by quadratures (in the generic case when no three of the vectors $J^i = (J_1^i,J_2^i,\dots,J_n^i)$, $1 \leqq i \leqq m$, are colinear); ii) they show how one can recover the scattering transform from (at least small) data given on certain (n+1)-dimensional surfaces (n+1 being the number of variables in Q); iii) they may indicate what other (possibly non-local) evolution equations could be solvable with the IST developed here; iv) they constitute in themselves a new class of multidimensional nonlinear systems of integro-differential equations which are linearizable.

2. The Case $\sigma_I \neq 0$.

We will denote by $k = (k_1,\ldots,k_n) = k_R + ik_I$ a point in $\mathbb{C}^n$ and will often write $f(k)$ instead of $f(k_R,k_I)$ for an arbitrary function of k_R and k_I.

As a first step in the $\bar\partial$ procedure we construct a family of solutions of $L_\sigma\psi = 0$ of the form $\psi = \mu(x_0,x,k)\exp[i\Sigma_{\ell=1}^n k_\ell(x_\ell-\sigma x_0 J_\ell)]$ with μ bounded; μ will then satisfy the equation

$$\frac{\partial\mu}{\partial x_0} + \sigma\Sigma_{\ell=1}^n J_\ell \frac{\partial\mu}{\partial x_\ell} + i\sigma\Sigma_{\ell=1}^n k_\ell[J_\ell,\mu] = Q\mu. \qquad (4)$$

The generalized eigenfunctions $\mu_\sigma = (\mu_\sigma^{ij})$ we will work with, are obtained by solving the integral equation $\mu_\sigma = I + \tilde{G}_\sigma(Q\mu_\sigma)$, i.e.

$$\mu_\sigma^{ij} = \delta_{ij} + \iint_{\mathbb{R}^{n+1}} G_\sigma^{ij}(x_0-y_0,x-y,k)(Q(y_0,y)\mu_\sigma(y_0,y,k))^{ij}dy_0dy, \qquad (5)$$

where the Green's function is given by

$$G_\sigma^{ij}(x_0,x,k) = \frac{-i}{(2\pi)^{n+1}} \iint_{\mathbb{R}^{n+1}} \frac{e^{i(x_0\xi_0 + x\cdot\xi)}}{\xi_0+\sigma\Sigma_{\ell=1}^n[J_\ell^i\xi_\ell+k_\ell(J_\ell^i-J_\ell^j)]} d\xi_0 d\xi. \qquad (6)$$

For brevity we will assume here that Q is such that this integral equation has a bounded solution μ_σ for all $k \in \mathbb{C}^n$.

G_σ can be computed explicitly by contour integration:

$$G_\sigma^{ij}(x_0,x,k) = \frac{\mathrm{sign}(\sigma_I J_1^i)}{2\pi i(x_1-\sigma J_1^i x_0)} e^{i\alpha_\sigma^{ij}(x_0,x,k)} \prod_{\ell=2}^n \delta(x_\ell - \frac{J_\ell^i}{J_1^i} x_1) \qquad (7)$$

with $$\alpha_\sigma^{ij}(x_0,x,k) = \sum_{\ell=1}^n \frac{J_\ell^i - J_\ell^j}{\sigma_I} (|\sigma|^2 x_0 k_{I_\ell} - \frac{x_\ell}{J_\ell^i}(\sigma k_\ell)_I). \qquad (8)$$

The next step is to express $\bar\partial\mu$ in terms of μ. We start by writing

$\frac{\partial G}{\partial \bar{k}_p}$ and hence $\frac{\partial \tilde{G}}{\partial \bar{k}_p}(Q\mu)$ as a superposition of exponentials:

$$\left(\frac{\partial \tilde{G}_\sigma}{\partial \bar{k}_p}(Q\mu_\sigma)\right)^{ij} = \frac{\bar{\sigma}(J_p^i - J_p^j)}{2i|\sigma_I|(2\pi)^n} \int_{\mathbb{R}^n} \sigma\left(\sum_{\ell=1}^n J_\ell^i \lambda_\ell\right) e^{i\beta_\sigma^{ij}(x_0,x,k,\lambda)} T_\sigma^{ij}(k,\lambda)d\lambda. \quad (9)$$

with $\beta_\sigma^{ij}(x_0,x,k,\lambda) = \alpha_\sigma^{ij}(x_0,x,k) + \sum_{\ell=1}^n (x_\ell - \sigma_R J_\ell^i x_0)\lambda_\ell$ and (10)

$$T_\sigma^{ij}(k,\lambda) = \iint_{\mathbb{R}^{n+1}} e^{-i\beta_\sigma^{ij}(y_0,y,k,\lambda)} (Q(y_0,y)\mu_\sigma(y_0,y,k))^{ij} dy_0 dy \ . \quad (11)$$

The calculation of $\bar{\partial}\mu$ is then based on the following crucial symmetry property of our Green's function:

$$e^{-i\beta_\sigma^{ij}(x_0,x,k,\lambda)} G_\sigma^{rj}(x_0,x,k) = G_\sigma^{ri}(x_0,x,\hat{k}_\sigma^{ij}(k,\lambda)) \text{ whenever } \Sigma J_\ell^i \lambda_\ell = 0; \quad (12)$$

here $\hat{k}_\sigma^{ij}(k,\lambda)$ is the point in $\mathbb{C}^n$ whose ℓ^{th} component is

$$(\hat{k}_\sigma^{ij}(k,\lambda))_\ell = \frac{J_\ell^j - J_\ell^i}{\sigma_I J_\ell^i}(\sigma k_\ell)_I + k_\ell + \lambda_\ell \ . \quad (13)$$

Once (12) has been established it can be shown (assuming that (5) admits no homogeneous solutions) that

$$\frac{\partial \mu_\sigma}{\partial \bar{k}_p}(x_0,x,k) = \sum_{i,j} \frac{\bar{\sigma}(J_p^i - J_p^j)}{2i|\sigma_I|(2\pi)^n} \int_{\mathbb{R}^n} \delta(\Sigma J_\ell^i \lambda_\ell) T_\sigma^{ij}(k,\lambda) e^{i\beta_\sigma^{ij}(x_0,x,k,\lambda)} \times$$

$$\times \mu_\sigma(x_0,x,\hat{k}_\sigma^{ij}(k,\lambda)) E_{ij} d\lambda; \quad (14)$$

(we have denoted by E_{ij} the m x m matrix with entries $E_{ij}^{rs} = \delta_{ir}\delta_{js}$). If we now fix all k_ℓ, $\ell \neq p$, and apply the (1-dimensional) inhomogeneous Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{|z'|=R} \frac{f(z')}{z'-z}\, dz' + \frac{1}{2\pi i} \iint_{|z'|\leq R} \frac{\frac{\partial f}{\partial \bar{z}}(z')}{z'-z}\, dz' \wedge d\bar{z}' \qquad (15)$$

to the k_p variable, we can convert $(14)_p$ to an integral equation: noting that $\mu(x_0,x,k) \sim I$ when $|k_p| \to \infty$ (and denoting $k' = (k_1,\ldots,k_p',\ldots,k_n)$) we have

$$\mu_\sigma(x_0,x,k) = I - \frac{i\bar{\sigma}}{|\sigma_I|(2\pi)^{n+1}} \sum_{i,j} (J_p^i - J_p^j) \iiint \frac{\delta(\Sigma J_\ell^i \lambda_\ell)}{k_p - k_p'} T_\sigma^{ij}(k',\lambda) e^{i\beta_\sigma^{ij}(x_0,x,k',\lambda)} \times$$

$$\times\, \mu_\sigma(x_0,x,\hat{k}^{ij}(k',\lambda)) E_{ij}\, d\lambda\, dk'_{R_p} dk'_{I_p}\,. \qquad (16)_p$$

(More generally, one can use (15) with $f(z) = \mu_\sigma(x_0,x,k+zv)$, $z \in \mathbb{C}$, with k fixed and with an arbitrary $v \in \mathbb{C}^n$ which is not perpendicular to any of the vectors $J^i - J^j$, $i \neq j$). The matrix-valued function $T_\sigma(k,\lambda)$ defined in (11) is our scattering data and (16) is the inverse scattering recipe for reconstructing μ from T. Once μ is found, the potential is easily recovered:

$$Q(x_0,x) = \frac{i\sigma}{\pi} [J_p, \iint \frac{\partial \mu_\sigma}{\partial \bar{k}_p} (x_0,x,k)\, dk_{R_p} dk_{I_p}]. \qquad (17)$$

On the other hand, given an arbitrary $T(k,\lambda)$, to apply the above inversion procedure we would first need to know that the equations $(14)_p$, $p = 1,2,\ldots,n$, are compatible; requiring that $\frac{\partial^2 \mu}{\partial \bar{k}_r \partial \bar{k}_p} = \frac{\partial^2 \mu}{\partial \bar{k}_p \partial \bar{k}_r}$ yields the following characterization equations for T:

$$L_{pr}^{ij}[T_\sigma] \doteq (J_p^i - J_p^j) \frac{\partial T_\sigma^{ij}}{\partial \bar{k}_r} - (J_r^i - J_r^j) \frac{\partial T_\sigma^{ij}}{\partial \bar{k}_p} + \frac{i\bar{\sigma}}{2\sigma_I} (J_p^i - J_p^j)(J_r^i - J_r^j)\left(\frac{1}{J_r^i} \frac{\partial T_\sigma^{ij}}{\partial \lambda_r} - \frac{1}{J_p^i} \frac{\partial T_\sigma^{ij}}{\partial \lambda_p}\right) =$$

$$= N_{pr}^{ij}[T_\sigma] \doteq \frac{i\bar{\sigma}}{2|\sigma_I|(2\pi)^n} \sum_{i'} [(J_p^{i'} - J_p^j)(J_r^i - J_r^{i'}) - (J_r^{i'} - J_r^j)(J_p^i - J_p^{i'})] \int \delta(\Sigma J_\ell^{i'} \lambda_\ell') T_\sigma^{i'j}(k,\lambda') \times$$

$$\times\, T_\sigma^{ii'}(\hat{k}^{i'j}(k,\lambda'), \lambda - \frac{J^{i'}}{J^i} \lambda')\, d\lambda'. \qquad (18)_{pr}^{ij}$$

For compatibility, $(18)^{ij}$ need only hold whenever $\Sigma J_\ell^i \lambda_\ell = 0$, however one may also verify that T_σ when given by (11) satisfies (18) everywhere.

It turns out to be very useful to recast (18) in integral form. It is enough to keep only the equations $(18)_{p1}$. We then look for a parametrization of the hyperplane $\{(k,\lambda) \in \mathbb{C}^n \times \mathbb{R}^n : \Sigma J^i_\ell \lambda_\ell = 0\}$ by new variables $(\chi, w_0, w) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$ so that, in the new coordinates $L^{ij}_{p1} = \frac{\partial}{\partial \bar{\chi}_p}$, $2 \leqq p \leqq n$ and $\beta^{ij}_\sigma(x_0, x, k, \lambda) = x_0 w_0 + x\cdot w$; these requirements determine (up to a translation of χ) the following (invertible) map:

$$k_\ell = (J^j_1 - J^i_1)\chi_\ell,\ \ell \geqq 2;\quad k_1 = \Sigma^n_{\ell=2}(J^i_\ell - J^j_\ell)\chi_\ell + \frac{1}{J^j_1 - J^i_1}\left(\frac{\bar{\sigma}}{|\sigma|^2} w_0 + \Sigma^n_{\ell=1} J^i_\ell w_\ell\right);$$

$$\lambda_\ell = \frac{(J^j_1 - J^i_1)(J^i_\ell - J^j_\ell)}{\sigma_I J^i_\ell}(\sigma\chi_\ell)_I + w_\ell,\ \ell \geqq 2;\ \lambda_1 = \Sigma^n_{\ell=2}\left[-\frac{(J^i_1 - J^j_1)(J^i_\ell - J^j_\ell)}{\sigma_I J^i_1}(\sigma\chi_\ell)_I - \frac{J^i_\ell}{J^i_1} w_\ell\right]. \tag{19}$$

To use (15) as before, we need the limit of T^{ij} for $|\chi_p|$ large (and χ_ℓ, $\ell \neq p$, w_0, w fixed); this turns out to depend on whether for some $r \neq i,j$ we have

$$(J^r_1 - J^j_1)(J^i_p - J^j_p) = (J^i_1 - J^j_1)(J^r_p - J^j_p)\ . \tag{20}$$

For brevity we consider only two cases (the only ones arising in the study of (2) - see the appendix): case I-relation (20) does <u>not</u> hold for any distinct i,j,r and any $p \neq 1$; and case II-relation (20) holds for all i,j,r,p (in other words, the vectors $J^1, \ldots, J^m$ all lie on the same line in $\mathbb{R}^n$). In the generic case I we have

$$\lim_{|\chi_p| \to \infty} T^{ij}_\sigma(\chi, w_0, w) = \hat{Q}^{ij}(w_0, w) \tag{21}$$

and $(18)^{ij}_{p1}$ becomes

$$I^{ij}_p[T_\sigma](\chi, w_0, w) \doteq T^{ij}_\sigma(\chi, w_0, w) - \frac{1}{\pi}\iint \frac{N^{ij}_{p1}[T_\sigma](\chi', w_0, w)}{\chi_p - \chi'_p}\, d\chi'_{R_p} d\chi'_{I_p} = \hat{Q}^{ij}(w_0, w), \tag{22$_I$}$$

where

$$\hat{Q}^{ij}(w_0,w) = \iint e^{-i(x_0w_0 + x\cdot w)}\, Q^{ij}(x_0,x)\, dx_0 dx \quad \text{and } \chi' = (\chi_2,\ldots,\chi_p',\ldots,\chi_n).$$

If (20) holds for some $r \neq j$ then (21) need not be true (see (7), (8), (11)). In case II we have $\frac{\partial T^{ij}}{\partial \bar{\chi}_p} \equiv 0$ for all p, $2 \leqq p \leqq n$; this, together with Liouville's theorem, allows us to replace $(22)_I$ by

$$T_\sigma^{ij}(\chi,w_0,w) = T_\sigma^{ij}(w_0,w). \tag{$(22)_{II}$}$$

In case I we conjecture (as in [2]) that the main condition needed to characterize the scattering data is that $I_p^{ij}[T_\sigma](\chi,w_0,w)$ be independent of χ and p and have suitable decay properties in (w_0,w); furthermore, given a T_σ which passes this admissibility test we can (re)construct a local potential Q simply as the inverse Fourier transform of I[T].

From $(22)_{II}$ we see that T^{ij} is completely determined by its values on the (n+1)-dimensional surface $\chi = \chi_0$; *the analogue* of this in case I is the following: given $T_\sigma^{ij}(\chi_0,w_0,w) = G^{ij}(w_0,w)$, $1 \leqq i,j \leqq m$ we have (from $(22)_I$)

$$T_\sigma^{ij}(\chi,w_0,w) = G^{ij}(w_0,w) + \frac{1}{\pi}\iint\left[\frac{N_{pl}^{ij}[T_\sigma](\chi',w_0,w)}{\chi_p-\chi_p'} - \frac{N_{pl}^{ij}[T_\sigma](\chi_0',w_0,w)}{\chi_{0p}-\chi_p'}\right]d\chi_{R_p}'\, d\chi_{I_p}' \tag{23}$$

which (at least for small G) could be solved to find T everywhere.

3. The case $\sigma = -1$.

If we formally substitute $\sigma = -1$ in (6) we find that, away from the hyperplanes $\Sigma_{ij} = \{k \in \mathbb{C}^n : \Sigma_{\ell=1}^n (J_\ell^i - J_\ell^j)k_{I_\ell} = 0\}$ the eigenfunction $\mu_{-1}(x_0,x,k)$ is well-defined and holomorphic. Thus it appears that the inverse problem for the hyperbolic system L_{-1} could be regarded as a Riemann-Hilbert problem with data on the hyperplanes Σ_{ij}, $1 \leqq i < j \leqq m$. We prefer to obtain an inversion procedure from our results for $\sigma_I \neq 0$. There seems to be little advantage in studying the limit of $\mu_\sigma(x_0,x,k)$ as $\sigma \to -1$ (it leads us back to an analysis of singularities on the hyperplanes Σ_{ij}); we work instead with the limit of $\mu_\sigma(x_0,x,k_R,\sigma_I k_I)$, with k_I now playing the role of a parameter. From (6) we find

$$\lim_{\sigma\to -1+i0} G_\sigma(x_0,x,k_R,\sigma_I k_I) = G_L(x_0,x,k_R,k_I)$$

$$= \frac{-i}{(2\pi)^{n+1}} \iint_{\mathbb{R}^{n+1}} \left\{ \frac{\theta(\Sigma_{\ell=1}^n[J_\ell^i\xi_\ell+(k_{R_\ell}-k_{I_\ell})(J_\ell^i-J_\ell^j)])}{\xi_0-\Sigma_{\ell=1}^n[J_\ell^i\xi_\ell+k_{R_\ell}(J_\ell^i-J_\ell^j)]+i0} + \frac{\theta(-\Sigma_{\ell=1}^n[J_\ell^i\xi_\ell+(k_{R_\ell}-k_{I_\ell})(J_\ell^i-J_\ell^j)])}{\xi_0-\Sigma_{\ell=1}^n[J_\ell^i\xi_\ell+k_{R_\ell}(J_\ell^i-J_\ell^j)]-i0} \right\} \times$$

$$\times\, e^{i(x_0\xi_0 + x\cdot\xi)} \, d\xi_0 d\xi, \tag{24}$$

with $\theta(\cdot)$ the Heaviside function; correspondingly, $\lim\limits_{\sigma\to -1+i0} \mu_\sigma(x_0,x,k_R,\sigma_I k_I) =$ $= \mu_L(x_0,x,k_R,k_I)$ where μ_L solves the integral equation $\mu_L = I + \tilde{G}_L(Q\mu_L)$. From (24) we see that $\mu_L(x_0,x,k_R,k_I)$ is a solution of

$$\frac{\partial\mu}{\partial x_0} - \Sigma_{\ell=1}^n J_\ell \frac{\partial\mu}{\partial x_\ell} - i\Sigma_{\ell=1}^n k_{R_\ell}[J_{\ell,\mu}] = Q\mu \tag{25}$$

for every value of the parameter k_I in $\mathbb{R}^n$. Our scattering data is now

$$T_L^{ij}(k_R,k_I,\lambda) = \iint_{\mathbb{R}^{n+1}} e^{-i\beta_L^{ij}(x_0,x,k_R,k_I,\lambda)} (Q(x_0,x)\mu_L(x_0,x,k_R,k_I))^{ij} dx_0 dx \tag{26}$$

with $\beta_L^{ij}(x_0,x,k_R,k_I,\lambda) = \Sigma_{\ell=1}^n[(J_\ell^i-J_\ell^j)(x_0 k_{I_\ell} - \frac{x_\ell}{J_\ell^i}(k_{R_\ell}-k_{I_\ell}))+(x_\ell+J_\ell^i x_0)\lambda_\ell]$. Taking limits in (14) we find the reconstruction equations for μ:

$$\mu_L(x_0,x,k_R,k_I) = I + \frac{i}{(2\pi)^{n+1}} \sum_{i,j}(J_p^i-J_p^j)\iiint\left[\frac{\theta(k_{I_p}-k'_{I_p})}{k_{R_p}-k'_{R_p}+i0} + \frac{\theta(k'_{I_p}-k_{I_p})}{k_{R_p}-k'_{R_p}-i0}\right]\delta(\Sigma J_\ell^i\lambda_\ell) \times$$

$$\times\ T_L^{ij}(k_R,k_I,\lambda)e^{i\beta_L^{ij}(x_0,x,k'_R,k'_I,\lambda)} \mu_L(x_0,x,\hat{k}_L^{ij}(k'_R,k'_I,\lambda))E_{ij}d\lambda dk'_{R_r} dk'_{I_p}, \tag{27}$$

where now $(\hat{k}_L^{ij}(k_R,k_I,\lambda))_{R_\ell} = \frac{J_\ell^j}{J_\ell^i} k_{R_\ell} + \frac{J_\ell^i - J_\ell^j}{J_\ell^i} k_{I_\ell} + \lambda_\ell$ and $(\hat{k}_L^{ij})_{I_\ell} = k_{I_\ell}$.

To write the characterization equations for T_L^{ij} we introduce new variables (suggested by the limit of (19)) $(\chi_R, \chi_I, w_0, w) \in \mathbb{R}^{3n-1}$ to parametrize the hyperplane $\Sigma J_\ell^i \lambda_\ell = 0$ in $\mathbb{R}^{3n}$ as follows:

$$k_{R_\ell} = (J_1^j - J_1^i)\chi_{R_\ell},\ \ell \geq 2; \quad k_{R_1} = \Sigma_{\ell=2}^n (J_\ell^i - J_\ell^j)\chi_{R_\ell} + \frac{1}{J_1^i - J_1^j}(w_0 - \Sigma_{\ell=1}^n J_\ell^i w_\ell)$$

$$k_{I_\ell} = (J_1^j - J_1^i)\chi_{I_\ell},\ \ell \geq 2; \quad k_{I_1} = \Sigma_{\ell=2}^n (J_\ell^i - J_\ell^j)\chi_{I_\ell} + \frac{1}{J_1^i - J_1^j} w_0 \tag{28}$$

$$\lambda_\ell = \frac{(J_1^j - J_1^i)(J_\ell^i - J_\ell^j)}{J_\ell^i}(\chi_{R_\ell} - \chi_{I_\ell}) + w_\ell,\ \ell \geq 2; \quad \lambda_1 = \Sigma_{\ell=2}^n \Big[\frac{(J_1^i - J_1^j)(J_\ell^i - J_\ell^j)}{J_1^i}(\chi_{R_\ell} - \chi_{I_\ell}) - \frac{J_\ell^i}{J_1^i} w$$

Then under the assumptions of case I in section 2, the limit of the equations $(22)_I$ yields:

$$T_L^{ij}(\chi_R, \chi_I, w_0, w) = \hat{Q}^{ij}(w_0, w) + \frac{1}{\pi}\iint \Big[\frac{\theta(\chi_{I_p} - \chi'_{I_p})}{\chi_{R_p} - \chi_{R_p}' + i0} + \frac{\theta(\chi'_{I_p} - \chi_{I_p})}{\chi_{R_p} - \chi_{R_p}' - i0}\Big] N_{p1}^{ij}[T_L](\chi', w_0, w)\, d\chi'_{R_p} d\chi'_{I_p}, \tag*{$(29)_I$}$$

with $N_{p1}[T_L]$ given by a slight modification of (18). In case II we have

$$T_L^{ij}(\chi_R, \chi_I, w_0, w) = T_L^{ij}(w_0, w). \tag*{$(29)_{II}$}$$

As in section 2, we can now use $(29)_I$ to characterize admissible T_L, (re)construct Q, as well as recover T_L from data given on χ_R = const., χ_I = const.

It should be pointed out that once the family of Green's functions G_L has been chosen, all the above results can be derived without recourse to our limiting arguments ($\nabla_{k_I}\mu_L$ can be expressed in terms of μ_L using the appropriate symmetry property of G_L and the analytic behaviour of μ_L for k_I large - needed to establish (27)- follows from (24); these analytic properties together with the compatibility requirements $\frac{\partial^2\mu}{\partial k_{I_r}\partial k_{I_p}} = \frac{\partial^2\mu}{\partial k_{I_p}\partial k_{I_r}}$ imply (29)).

4. Relation between T_L and the Scattering Operator ($\sigma = -1$)

To fix notation we sketch an elementary definition of the scattering operator associated with L_{-1}. When $Q \equiv 0$, given $f : \mathbb{R}^n \to \mathbb{R}^m$, the solution of the Cauchy problem $L_{-1}u(x_0,x) = 0$, $u(0,x) = f(x)$ is $u^i(x_0,x) = f^i(x_1+x_0J_1^i,\dots,x_n+x_0J_n^i)$, $1 \leqq i \leqq m$, which we write as $u(x_0,x)=f(x+x_0J)$. When Q is, say, smooth and of compact support, given any (reasonable) $f : \mathbb{R}^n \to \mathbb{R}^m$ there is a unique u solution of $L_{-1}u = 0$ with $u(x_0,x) = f(x+x_0J)$ for $x_0 << 0$; furthermore there is a unique g such that $u(x_0,x) = g(x+x_0J)$ when $x_0 >> 0$. We write $g = Sf$. On the Fourier transform side S can be written as

$$\hat{S}f(\xi) = f(\xi) + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} S(\xi,k_R)\hat{f}(k_R)dk_R. \tag{30}$$

The question we would like to address is how to recover T_L (and hence Q) given $S(\xi,k_R)$. To relate T_L and $S(\xi,k_R)$ it turns out that we need to relate μ_L and the eigenfunction $\mu(x_0,x,k_R)$ corresponding to the "Volterra" Green's function

$$G^{ij}(x_0,x,k_R) = \theta(x_0)\exp[-i\Sigma_{\ell=1}^n(x_\ell+x_0J_\ell^j)k_{R_\ell}]\prod_{\ell=1}^n\delta(x_\ell+x_0J_\ell^i). \tag{31}$$

We start with the identity

$$\mu_L - \mu = (\tilde{G}_L-\tilde{G})(Q\mu_L) + \tilde{G}(Q(\mu_L-\mu)), \tag{32}$$

write $G_L^{ij}-G^{ij}$ as a superposition of $\exp(i\beta_L^{ij})$ and use a suitable symmetry property of G. The main result is the following <u>linear</u> equation for T_L given S:

$$T_L^{ij}(k_R,k_I,\lambda) = S^{ij}(\hat{k}_R^{ij}(k_R,k_I,\lambda),k_R) - \frac{1}{(2\pi)^n}\sum_{i'}\int_{\mathbb{R}^n}\theta(\sum_{\ell=1}^n J_\ell^{i'}\lambda_\ell)\times$$

$$\times S^{ii'}(\hat{k}_R^{ij}(k_R,k_I,\lambda),\hat{k}_R^{i'j}(k_R,k_I,\lambda'))T_L^{i'j}(k_R,k_I,\lambda')d\lambda', \tag{33}$$

where $\hat{k}_R^{ij}(k_R,k_I,\lambda)$ stands for the real part of $\hat{k}_L^{ij}$.

5. Applications to Nonlinear Equations

The equations (2) are the compatibility conditions (cf. [ ]) for the Lax pair:

$$L_\sigma \psi = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial t} + \Sigma_{\ell=1}^{n} B_\ell \frac{\partial \psi}{\partial x_\ell} = A\psi; \tag{34}$$

the matrices B_ℓ, $1 \leqq \ell \leqq n$, are constant real diagonal and $A^{ij}(t,x_0,x) = \frac{1}{\sigma} a_{ij} Q^{ij}(t,x_0,x)$ with a_{ij} given by (3). The restrictions imposed by (3) on the matrices J_ℓ, $1 \leqq \ell \leqq n$, are discussed in the appendix. To find the time dependence of the scattering data corresponding to (2), we set $\psi = \mu \exp[i\, \Sigma_{\ell=1}^{n} k_\ell (x_\ell - \sigma x_0 J_\ell - tB_\ell)]$; then μ satisfies (4) as well as

$$A\mu \doteq \frac{\partial \mu}{\partial t} + \Sigma_{\ell=1}^{n} B_\ell \frac{\partial \mu}{\partial x_\ell} + i\Sigma_{\ell=1}^{n} k_\ell [B_\ell, \mu] - A\mu = 0. \tag{35}$$

Applying the operator A to both sides of the equation (14) we find (when $\sigma_I \neq 0$)

$$\frac{\partial T_\sigma^{ij}}{\partial t}(t,k,\lambda) = i\Sigma_{\ell=1}^{n} [B_\ell^j k_\ell - B_\ell^i \hat{k}_\ell^{ij}(k,\lambda)] T_\sigma^{ij}(t,k,\lambda). \tag{36}$$

For the case $\sigma = -1$ the equations (obtained as limits of (36) or by a parallel derivation) are

$$\frac{\partial T_L^{ij}}{\partial t}(t,k_R,k_I,\lambda) = i\Sigma_{\ell=1}^{n} [B_\ell^j k_{R_\ell} - B_\ell^i \hat{k}_{R_\ell}^{ij}(k_R,k_I,\lambda)] T_L^{ij}(t,k_R,k_I,\lambda). \tag{37}$$

Thus, when $\sigma = -1$, we can apply the inverse scattering procedure together with (37) to construct the solution to the Cauchy problem for (2). Note that $T_L(t,k_R,k_I,\lambda)$ as given by (37) satisfies the characterization equations if $T_L(0,k_R,k_I,\lambda)$ does.

When $\sigma_I \neq 0$ the Cauchy problem for (2) is ill-posed; however (by analogy to the corresponding linear problem) we can use inverse scattering to solve (2) as follows: given $Q(0,x_0,x)$ it can be decomposed into $Q_+(0,x_0,x) + Q_-(0,x_0,x)$

where $Q_+(0,x_0,x)$ extends to a function $Q_+(t,x_0,x)$ satisfying (2) in the half-space $t > 0$, while $Q_-(0,x_0,x)$ extends to a function satisfying (2) in the half-space $t < 0$. Assume for simplicity that $\sigma_I a_{ij} > 0$ for all $i \neq j$. Given Q define $Q_{\pm}$ by $\hat{Q}_{\pm}(0,w_0,w) = \theta(\mp w_0)\hat{Q}(0,w_0,w)$. If T_+ is the scattering transform of Q_+ then from the direct problem we find $T_+^{ij}(0,\chi,w_0,w) = 0$ for $w_0 > 0$; thus for $t > 0$ we can define (see (36)) $T_+^{ij}(t,\chi,w_0,w)$ by

$$T_+^{ij}(t,\chi,w_0,w) = \exp[it\Sigma_{\ell=1}^{n}(B_\ell^j k_\ell - B_\ell^i \hat{k}_\ell^{ij})]T_+^{ij}(0,\chi,w_0,w) = \text{(see (3), (13) and (19))}$$

$$= \exp[it(\frac{a_{ij}}{\sigma} w_0 + \Sigma_{\ell=1}^{n}(a_{ij}J_\ell^i - B_\ell^i)w_\ell)]T_+^{ij}(0,\chi,w_0,w). \qquad (38)$$

Since the expression in the exponential does not depend on χ and since its real part is nonpositive if $t > 0$, $T_+^{ij}(t,\chi,w_0,w)$ satisfies the characterization equations (29) so inverse scattering should yield the desired potential $Q_+(t,x_0,x)$; similarly we can construct $Q_-(t,x_0,x)$ solution of (2) for $t < 0$.

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Appendix

We need to find the restrictions imposed on the choice of matrices J_ℓ, $1 \leqq \ell \leqq n$, by the existence of (a_{ij}) and B_ℓ, $1 \leqq \ell \leqq n$ satisfying (3).

If (3) holds then the matrix (a_{ij}) is symmetric and

$$a_{ip} - a_{pj} = (a_{ij}-a_{pj}) \frac{J_\ell^j - J_\ell^i}{J_\ell^p - J_\ell^i} \text{ for every } \ell \text{ and every } i,j,p \text{ distinct} . \tag{A1}$$

(Conversely, if (A1) holds with (a_{ij}) symmetric then B_ℓ, $1 \leqq \ell \leqq n$ can be found so that (3) is satisfied.) Note that if $a_{ip} \neq a_{pj}$, (A1) forces J^i, J^j, J^p to be colinear. There are two cases:

I $\quad a_{ip} = a_{pj}$ for all i,j,p distinct. Then (A1) puts no restriction on J_ℓ; in particular they can be chosen so that (20) does not hold for any distinct i,j,r and $p \neq 1$. The system (2) is linear in this case.

II $\quad$ For some i_0,j_0,p_0 distinct $a_{i_0p_0} \neq a_{p_0j_0}$. We show that in this case the vectors $J^1,\ldots,J^m$ must <u>all</u> be colinear. From (A1) we already know that J^{i_0}, J^{j_0}, J^{p_0} are colinear. For any $r \neq i_0,j_0,p_0$ one of the following must be true

$$\text{(i)} \quad a_{i_0r} \neq a_{rj_0}, \qquad \text{(ii)} \quad a_{ri_0} \neq a_{i_0p_0} , \qquad \text{(iii)} \quad a_{rj_0} \neq a_{j_0p_0} \tag{A2}$$

(for if not $a_{i_0p_0} = a_{ri_0} = a_{rj_0} = a_{p_0j_0}$ contradicting our assumption). In either of the possibilities (A2) J^r will be on the line passing through J^{i_0}, J^{p_0}, J^{j_0}; this will be true for any r, $1 \leqq r \leqq m$. (Conversely, given $J^1,J^2,\ldots,J^m$ colinear with $J_\ell^i \neq J_\ell^j$ we can construct (a_{ij}) symmetric satisfying II and (A1)).

It follows that whenever (2) is not linear, the matrix having $J^1,J^2,\ldots,J^m$ as rows has rank at most 2; thus if $n \geq 3$ its columns (the diagonals of the matrices J_ℓ in (1)) must be linearly dependent and then the inverse scattering problem for L_σ can also be solved by reducing it to a lower dimensional one. On the other hand, since the characterization equations are trivial (i.e. $N(T) = 0$) in this case, it seems reasonable to expect that other (possibly non-local) nonlinear equations can be found which would be compatible with $(22)_{II}$.

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